Parity of the Partition Function and Rademacher's Exact Formula

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$$p(n) = p(n-1) + p(n-2) - \cdots$$

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi \sqrt{\frac{2n}{3}}\right)$$

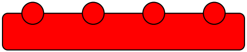
$$p(n) = \sum_{k=1}^{\infty} R_k(n),$$

$$R_k(n) = \frac{A_k(n)\sqrt{k}}{\pi\sqrt{2}} \frac{d}{dn} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}}$$

Overview

1. Introduction
2. Goal of this talk
3. Euler' Pentagonal Number Theorem and Recurrence
4. Parity of $p(n)$
5. Asymptotic formula for the partition function
6. Rademacher's Exact Formula for $p(n)$

Child playing with LEGO bricks



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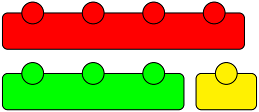

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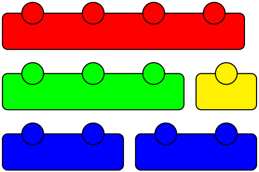

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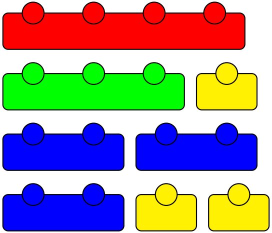

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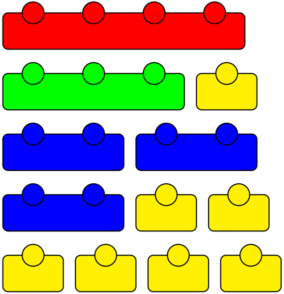

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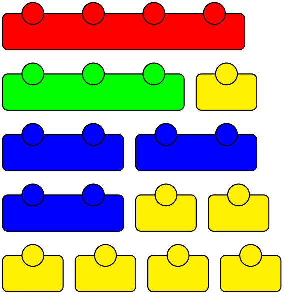
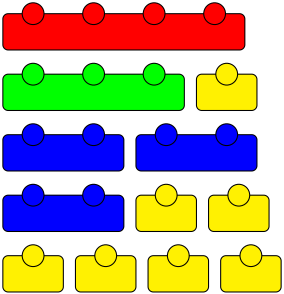


We say $p(4) = 5$.

Child playing with LEGO bricks



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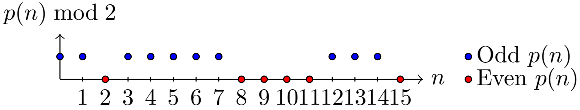
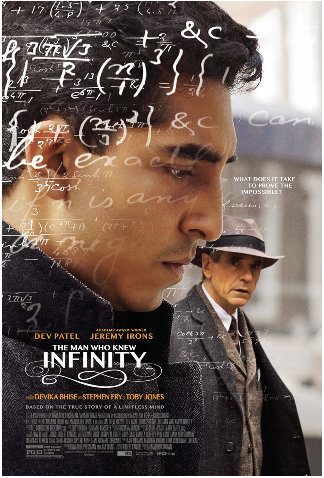
Definition (Partition function)

A partition of $n \in \mathbb{N}$ is a finite non-increasing sequence of non-negative integers whose sum is n. The number of such partitions is denoted $p(n)$, set $p(0) := 1$.

Project Goals

- The exact formula for the partition function (Hardy – Ramanujan – Rademacher approximation formula)
- $p(n)$ takes both even and odd values infinitely often.





Can we count the "uncountable"?

- $p(2) = 2$
- $p(4) = 5$
- $p(8) = 22$
- $p(16) = 231$
- $p(32) = 8,349$
- $p(64) = 1,741,630$
- $p(128) = 4,351,078,600$
- $p(256) = 365,749,566,870,782$
- $p(512) = 4,453,575,699,570,940,947,378$

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I dare you to count $p(200)$

Generating Function of $p(n)$

We now turn to generating functions, a central tool in partition theory. These functions encode infinite combinatorial data into a compact analytic form.

Theorem (Euler's Theorem)

$$\boxed{\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}} \quad \textit{for } |q| < 1.$$

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Proof.

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} q^{mn} = \left(\sum_{m=0}^{\infty} q^m\right)\left(\sum_{m=0}^{\infty} q^{2m}\right)\left(\sum_{m=0}^{\infty} q^{3m}\right)\cdots =$$

$$= \sum_{\substack{k \geqslant 0 \\ \alpha_1, \ldots, \alpha_k \geqslant 0}} q^{\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + k\alpha_k} = \sum_{\substack{k \geqslant 0 \\ \lambda = 1^{\alpha_1} 2^{\alpha_2} \ldots (k-1)^{\alpha_{k-1}} k^{\alpha_k} \vdash n}} q^{\lambda} = \sum_{n=0}^{\infty} p(n)\, q^n$$

$\square$

Euler' Pentagonal Number Theorem and Recurrence

Applying the Jacobi Triple Product, we get

$$\boxed{\prod_{m=1}^{\infty}(1-q^m)=\sum_{k=-\infty}^{\infty}(-1)^k\,q^{\frac{k(3k-1)}{2}}} \qquad \text{Euler's Pentagonal Number Theorem}$$

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Euler's Pentagonal Number Theorem

We know that:

$$\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{m=1}^{\infty}\frac{1}{1-q^m} \implies \left(\prod_{m=1}^{\infty}(1-q^m)\right)\left(\sum_{n=0}^{\infty} p(n)\, q^n\right) = 1 \implies$$

$$\implies \left(\sum_{r=-\infty}^{\infty}(-1)^r\, q^{\frac{r(3r-1)}{2}}\right)\left(\sum_{n=0}^{\infty} p(n)\, q^n\right) = 1 \implies$$

$$\implies \left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots\right)$$

$$\left(p(0) + p(1)q + p(2)q^2 + p(3)q^3 + p(4)q^4 + p(5)q^5 + \cdots\right) = 1 \implies$$

$$\implies p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7)$$

$$- p(n-12) - p(n-15) + p(n-22) + p(n-26) + \cdots = 0$$

$$\implies p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) +$$

$$+ p(n-12) + p(n-15) - p(n-22) - p(n-26) + \cdots .$$

Euler's Recurrence

Theorem (Euler (1700s))

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \\ + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \cdots .$$

or equivalently

$$\boxed{p(n) = \sum_{\substack{k \geqslant 1 \\ \frac{k(3k-1)}{2} \leqslant n}} (-1)^{k-1} \left[p\left(n - \frac{k(3k-1)}{2}\right) + p\left(n - \frac{k(3k+1)}{2}\right) \right]}$$

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Remark

The first 200 values were famously computed this way in 1915.

$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, \overline{490}, 627, 792, 1002,$
$1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, \ldots\ldots\ldots\ldots, p(200) = 3,972,999,029,388$

How often is $p(n)$ even?

Let $\text{Prop}_2(N) := \dfrac{\#\{0 \leqslant n \leqslant N : p(n) \text{ is even}\}}{N}$ the proportion of the first N values that are even.

N	$\text{Prop}_2(N)$
200,000	0.5012...
600,000	0.5000...
1,000,000	0.5004...
∞	$\frac{1}{2}$?

Table: Proportion of even values

Conjecture

Half of the partition numbers are even.

Theorem (O. Kolberg [1] (1959))

$p(n)$ takes both even and odd values infinitely often.

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Proof.

Suppose for contradiction that: $p(n) = \text{odd}$ for finitely many n's, then

$$p(n) = \text{even for infinitely many } n'\text{s} \iff p(n) \equiv 0 \pmod 2, \ \forall n \geqslant m \quad \text{for some } m \in \mathbb{N}.$$

Take $n := \frac{m(3m-1)}{2} > m$, then we have: $n - \frac{k(3k \pm 1)}{2} > m$ for all $k = 0, 1, 2, \ldots, m-1$, thus $p\left(n - \frac{k(3k \pm 1)}{2}\right) \equiv 0 \pmod 2$, so

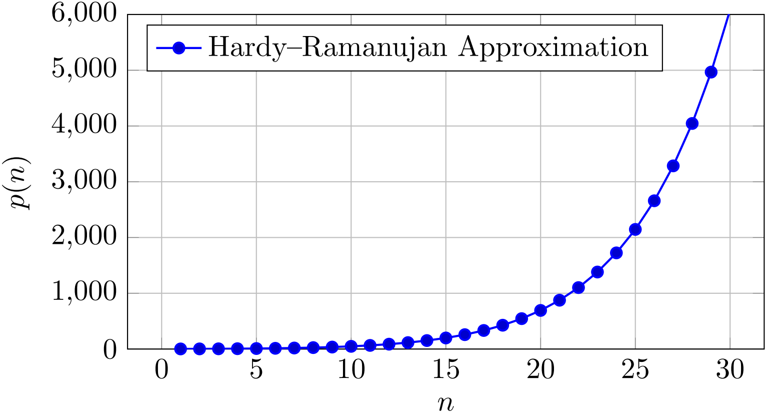
$$p\left(\frac{m(3m-1)}{2}\right) =$$

$$= p\left(\frac{m(3m-1)}{2} - 1\right) + p\left(\frac{m(3m-1)}{2} - 2\right) - p\left(\frac{m(3m-1)}{2} - 5\right) -$$
$$- p\left(\frac{m(3m-1)}{2} - 7\right) + p\left(\frac{m(3m-1)}{2} - 12\right) + p\left(\frac{m(3m-1)}{2} - 15\right) -$$
$$- p\left(\frac{m(3m-1)}{2} - 22\right) - p\left(\frac{m(3m-1)}{2} - 26\right) + \cdots \mp p(2m-1) \pm p(0) \rightarrow\leftarrow$$

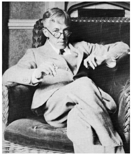
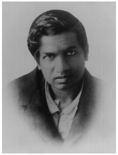
□

Growth of $p(n)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176



Historical Line: Partition Function

1918

Hardy–Ramanujan
Asymptotic formula for $p(n)$:
(circle method)

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

$$p(n) = \sum_{k<\alpha\sqrt{n}} P_k(n) + O(n^{-1/4})$$

1937

Lehmer

Shows $\sum_{k=1}^{\infty} P_k(n)$ diverges

1937-1943

Rademacher
Exact convergent series:

$$p(n) = \sum_{k=1}^{\infty} R_k(n)$$

with the remainder after $N \asymp \sqrt{n}$ error term= $O(n^{-1/4})$

Rademacher's Exact Formula for $p(n)$

If $n \geqslant 1$, the partition function $p(n)$ is represented by the convergent series

$$p(n) = \sum_{k=1}^{\infty} R_k(n)$$

where

$$R_k(n) = \frac{\sqrt{k}}{\pi\sqrt{2}}\, A_k(n)\, \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right)$$

$$A_k(n) = \sum_{\substack{0\leqslant h\leqslant k \\ \gcd(h,k)=1}} e^{\pi i\, s(h,k) - \frac{2\pi i n h}{k}} \quad , \qquad s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right).$$

Rademacher's series is also rapidly convergent, with its first term providing the dominant term in the growth rate for $p(n)$.

Functional Equation of the Dedekind Eta Function

Rademacher's series arises by applying contour integration along $P(N)$ together with the functional equation of the Dedekind eta function. For the partition generating function

Theorem

Let $F(q) = \sum_{n=0}^{\infty} p(n)\, q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}$ *and let*

$$w := \exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right), \qquad w' := \exp\left(\frac{2\pi i H}{k} - \frac{2\pi}{z}\right)$$

where $\Re(z) > 0,\ 0 \leqslant h \leqslant k,\ \gcd(h,k) = 1,$ *and* $hH \equiv -1 \pmod{k}$. *Then*

$$F(w) = e^{\pi i\, s(h,k)} \left(\frac{z}{k}\right)^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F(w').$$

Notice that $F(q)$ has poles at the roots of unity. When $|z|$ is small, then w is close to a root of unity and w' is close to 0, thus $F(w') \approx 1$. So when w is close to a singularity point, then $F(w) \sim e^{\pi i\, s(h,k)} \left(\frac{z}{k}\right)^{\frac{1}{2}} e^{\frac{\pi}{12z}}$. **This asymptotic behavior is essential when applying Cauchy's theorem along Rademacher's contour**.

From Cauchy's Formula to a Modular Contour

By Cauchy's formula, for any closed contour C inside the unit disk,

$$p(n) = \frac{1}{2\pi i} \oint_C \frac{F(q)}{q^{n+1}}\, dq, \qquad F(q) = \sum_{n=0}^{\infty} p(n)\, q^n = \prod_{m \geqslant 1} \frac{1}{1-q^m}.$$

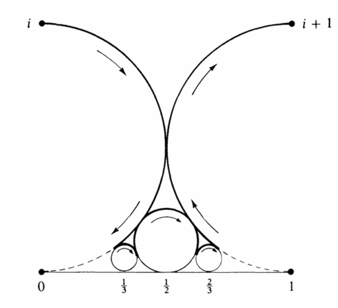
Since F is analytic in $|q| < 1$ except at roots of unity on $|q| = 1$, *any* simple closed curve strictly inside the unit circle is allowed.

Change of variables. Set $q = e^{2\pi i \tau}$, $dq = 2\pi i\, e^{2\pi i \tau}\, d\tau$. Under this substitution, the contour C becomes any path from τ_0 to $\tau_0 + 1$ satisfying $|e^{2\pi i \tau_0}| < 1$. For convenience we choose $\tau_0 = i$, although the specific choice is irrelevant. Then

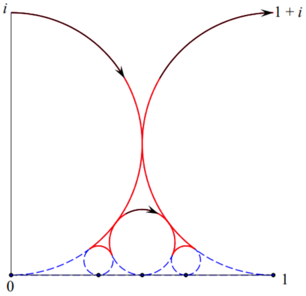
$$p(n) = \int_{i \to 1+i} F\left(e^{2\pi i \tau}\right)\, e^{-2\pi i n \tau}\, d\tau,$$

where the integral runs over *some path* connecting i to $1 + i$ in the strip $\mathrm{Im}(\tau) > 0$.

Key question: **Which path should we take?**



i
i + 1
0
1/3
1/2
2/3
1



i
1 + i
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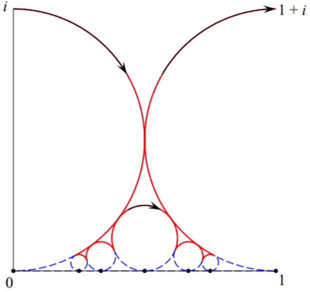
1



i
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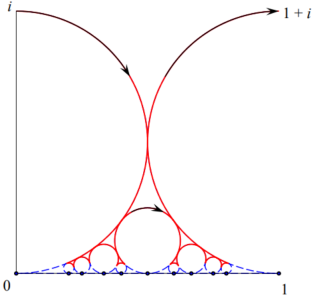


i
1 + i
0
1

Contour Integration

Using Rademacher's contour $P(N)$, we begin with

$$p(n) = \int_{i\to 1+i} F\left(e^{2\pi i\tau}\right) e^{-2\pi i n\tau}\, d\tau = \sum_{k=1}^{N} \sum_{\substack{0\leqslant h<k \\ \gcd(h,k)=1}} \int_{\gamma(h,k)} F\left(e^{2\pi i\tau}\right)\; e^{-2\pi i n\tau}\, d\tau.$$

Change of variables: Set $z = -ik^2\left(\tau - \frac{h}{k}\right)$ so that the functional equation for F can be applied.

- This maps each Ford circle $C(h,k)$ to a circle of radius $1/2$ centered at $1/2$.
- The arc $\gamma(h,k)$ becomes an arc from $z_1(h,k)$ to $z_2(h,k)$ on this new circle.

Under this change of variables, let $w := \exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right)$, $w' := \exp\left(\frac{2\pi i H}{k} - \frac{2\pi}{z}\right)$ so that the functional equation gives us

$$p(n) = \sum_{k=1}^{N} \sum_{\substack{0\leqslant h<k \\ \gcd(h,k)=1}} \frac{i}{k^2} \int_{z_1(h,k)}^{z_2(h,k)} w^{-n} F(w)\, dz =$$

$$= \sum_{k=1}^{N} \sum_{\substack{0\leqslant h<k \\ \gcd(h,k)=1}} \frac{i}{k^{5/2}} e^{\pi i\, s(h,k) - \frac{2\pi i n h}{k}} \int_{z_1(h,k)}^{z_2(h,k)} z^{1/2} \exp\left(\frac{\pi}{12z} - \frac{2\pi z\left(n - \frac{1}{24}\right)}{k^2}\right) F(w')\, dz.$$

Completing the Integral

Since $F(w') \approx 1$ on these arcs, the factor $F(w')$ may be neglected, and the resulting error is $O(N^{-1/2})$. Thus, letting $N \to \infty$ drives the error to zero. So

$$p(n) = \sum_{k=1}^{N} \sum_{\substack{0 \leqslant h < k \\ \gcd(h,k)=1}} \frac{i}{k^{5/2}} e^{\pi i\, s(h,k) - \frac{2\pi i n h}{k}} \int_{z_1(h,k)}^{z_2(h,k)} z^{1/2} \exp\left(\frac{\pi}{12z} - \frac{2\pi z\left(n - \frac{1}{24}\right)}{k^2}\right) dz + O(N^{-1/2}).$$

Idea: Instead of integrating only over the arc from z_1 to z_2, extend the integral to the **entire circle** K^- : $\left|z - \frac{1}{2}\right| = \frac{1}{2}$, traversed clockwise.

We then subtract the small connecting arcs: $\displaystyle\int_{z_1}^{z_2} = \oint_{K^-} - \int_{0}^{z_1} - \int_{z_2}^{0}$.

Inserting this into the expression for $p(n)$ gives

$$p(n) = \sum_{k=1}^{N} \sum_{\substack{0 \leqslant h < k \\ \gcd(h,k)=1}} \frac{i}{k^{5/2}} e^{\pi i\, s(h,k) - \frac{2\pi i n h}{k}} \left(\oint_{K^-} - \int_{0}^{z_1} - \int_{z_2}^{0}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{2\pi z\left(n - \frac{1}{24}\right)}{k^2}\right) dz + O(N^{-1/2}).$$

Finishing the proof

It can be shown that the sums of the integrals $\int_0^{z_1}$ and $\int_{z_2}^{0}$ contribute only $O(N^{-1/2})$,

so the only main term comes from the **single integral over** K^-, which is independent of h.
Thus,

$$p(n) = \sum_{k=1}^{N} \frac{i}{k^{5/2}} \left(\sum_{\substack{0 \leqslant h < k \\ \gcd(h,k)=1}} e^{\pi i\, s(h,k) - \frac{2\pi i n h}{k}} \right) \oint_{K^-} z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{2\pi z \left(n - \frac{1}{24}\right)}{k^2} \right) dz$$
$$+ O(N^{-1/2}).$$

The inner sum is precisely the Kloosterman-like sum $A_k(n)$, and the integral over K^- evaluates exactly in terms of the Bessel function $I_{3/2}$, which reduces to a hyperbolic sine.
Taking $N \to \infty$ eliminates the error and yields Rademacher's exact convergent series for $p(n)$. □

Application of Rademacher's formula

This remarkable identity provides not only a theoretical formula for $p(n)$ but also a method for relatively rapid computation. For example, $p(200) = 3,972,999,029,388$.

Using only the first eight terms of the series $\sum_{k=1}^{\infty} R_k(n)$ gives

$$
\begin{aligned}
R_1(200) &= & 3,972,998,993,185.896 \\
R_2(200) &= & +36,282.978 \\
R_3(200) &= & -87.555 \\
R_4(200) &= & +5.147 \\
R_5(200) &= & +1.424 \\
R_6(200) &= & +0.071 \\
R_7(200) &= & +0.000 \\
R_8(200) &= & +0.043 \\
\hline
& & 3,972,999,029,388.004
\end{aligned}
$$

which agrees with the true value of $p(200)$ to within 0.004. Because the truncation error can be bounded explicitly, we can compute exact values of $p(n)$ directly from the series.

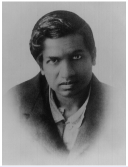
Why Do Partitions Matter? & Anticipators of the Theory

Applications and Connections

- Modular & automorphic forms — generating functions, congruences, Galois representations, and the "circle method".
- Symmetric functions & representation theory — Young diagrams, Schur/Hall-Littlewood bases, hook-length formula, Frobenius characteristic, S_n characters, ...
- $q-$series & identities — Rogers–Ramanujan, Andrews–Gordon, Bailey chains, product–sum transformations, ...
- Geometry & physics — Hilbert schemes of points, Donaldson/Gromov–Witten/BPS state counts, VOA/partition functions in topological strings.
- Probability & statistical mechanics — Plancherel measure, limit shapes, plane partitions, Bose–Einstein combinatorics.

Anticipators of the Theory

- (Hardy–Ramanujan, Rademacher) "Circle method"
- (Deligne–Serre) Galois representations
- (Mordell–Hecke) Hecke operators
- (Erdős) Probabilistic number theory
- (Deligne–Weil) Ramanujan–Peterson and Weil conjectures



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Thank you! Questions?

Jacobi Triple Product

$$\sum_{m=-\infty}^{\infty} z^m\, q^{\frac{m(m+1)}{2}} = \prod_{n=1}^{\infty} (1-q^n)\,(1+z\,q^n)\left(1+\frac{q^{n-1}}{z}\right) \quad \text{for } |q|<1,\ z\in\mathbb{C}\setminus\{0\}.$$

The Laurent series of $G(z) = \prod_{n\geqslant 1}(1+z\,q^n)\left(1+\frac{q^{n-1}}{z}\right)$ at $z=0$ is

$G(z) = \sum_{n=-\infty}^{\infty} A_n(q)\,z^n$. Shifting $z\mapsto zq$, $G(zq) = \prod_{n\geq 1}(1+zq^{n+1})\left(1+\frac{q^{n-2}}{z}\right) = \frac{1}{zq}G(z)$.

Comparing coefficients of z^n gives $q^n A_n(q) = q^{-1}A_{n+1}(q) \Longrightarrow A_n(q) = q^{n(n+1)/2}A_0(q)$.

The constant term is $A_0(q) = \prod_{n\geqslant 1}\frac{1}{1-q^n}$, because we write

$G(z) = (1+zq)(1+zq^2)(1+zq^3)\cdot\ldots\left(1+\frac{1}{z}\right)\left(1+\frac{q}{z}\right)\left(1+\frac{q^2}{z}\right)\left(1+\frac{q^3}{z}\right)\cdot\ldots$, so we need exactly the same number(say s) of second terms selected from each product(parenthesis) contained in $G(z)$, so $q^s\cdot q^{a_1+a_2+\cdots+a_s}\cdot q^{b_1+b_2+\cdots+b_s}$ where

$a_1>a_2>\cdots>a_s\geqslant 0$, $b_1>b_2>\cdots>b_s\geqslant 0$, so we have $s+\sum_{i=1}^{s}a_i+\sum_{i=1}^{s}b_i$, i.e.

$$A_0(q) = \sum_{m\geqslant 0} p(m)\,q^m = \prod_{n\geqslant 1}\frac{1}{1-q^n}.$$